Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE STRAIGHT LINE SOLUTIONS OF THE PROBLEM OF N BODIES*

By F. R. Moulton

1. Introduction.

In an elegant prize memoir,† Lagrange proved that there are, for any three finite masses attracting one another according to the Newtonian law, four distinct configurations such that, under proper initial projections, the *ratios of the mutual distances remain constants.* In all the configurations the bodies describe similar conic sections with respect to the center of mass of the system, the simplest case being that in which the orbits are circular. In three of the four solutions the masses lie always in a straight line, and in the fourth they remain at the vertices of an equilateral triangle.

In this paper two closely related problems are treated:

I. The number of straight line solutions is found for n arbitrary positive masses‡; that is, the ratios of the distances are determined so that under proper initial projections the bodies will always remain collinear. This is the direct generalization to the problem of n bodies of Lagrange's straight line solutions. The method of Lagrange is not suitable for treating the general case. Below the problem is formulated so that it is solved by finding the total number of real solutions of n simultaneous fractional algebraic equations. From the mathematical standpoint the interest in this part of the paper centers in the algebraical problem of finding the number of real solutions of a formidable looking set of simultaneous equations.

II. The second problem is that of determining, when possible, n masses such that if they are placed at n arbitrary collinear points, they will, under proper initial projection, always remain in a straight line. If n is even and the linear dimensions of the orbits are given, it is proved that the n masses are in general uniquely determined; and that if n is odd the coördinates of the n points must satisfy one algebraic relation, after which, choosing any one of the

---

*The greater part of this paper was written in 1900 and was presented to the Chicago Section of the Am. Math. Soc., Dec. 28, 1900. See *Bull. Am. Math. Soc.*, vol. 7 (1900–1901), p. 249. It has been recast and put in form for publication by the author as a Research Associate of the Carnegie Institution of Washington.

†Lagrange, *Collected Works*, vol. 6, pp. 229–324.

‡Treated briefly by Lehmann-Filhès in *Astronomische Nachrichten*, vol. 127 (1891), no. 033.

masses arbitrarily, the remaining $n-1$ are uniquely determined. This problem is in a certain sense the converse of the preceding.

I. DETERMINATION OF THE POSITIONS WHEN THE MASSES ARE GIVEN.

2. The Equations Defining the Solutions. Let the origin of coördinates be taken at the center of gravity of the system. This point and the line of initial projection of any mass determine a plane. All the other masses must be projected in this plane, for otherwise they would not be collinear at the end of the first element of time. All the bodies being initially in a line and projected in the same plane, they will always remain in this plane. Consequently if solutions exist in which the n masses are always in a straight line, the orbits are plane curves.

Let the plane of motion be the $\xi\eta$ — plane. Let the masses be denoted by $m_1, m_2, \cdots, m_n$, and their respective coördinates by $(\xi_1, \eta_1), (\xi_2, \eta_2), \cdots, (\xi_n, \eta_n)$. Then, choosing the units so that the Gaussian constant is unity, the differential equations of motion are

$$(1) \quad \left\{ \begin{aligned} &\frac{d^2\xi_i}{dt^2} = \frac{1}{m_i} \frac{\partial U}{\partial \xi_i}, \quad (i = 1, \cdots, n), \\ &\frac{d^2\eta_i}{dt^2} = \frac{1}{m_i} \frac{\partial U}{\partial \eta_i}, \\ &U = \tfrac{1}{2} \sum_{j=1}^{n} \sum_{k=1}^{n} \frac{m_j \, m_k}{r_{jk}}, \quad (j \neq k), \\ &r_{jk} = \sqrt{(\xi_j - \xi_k)^2 + (\eta_j - \eta_k)^2}. \end{aligned} \right.$$

Equations (1) admit the integral of areas

$$(2) \quad \left\{ \begin{aligned} &\sum_{i=1}^{n} m_i \left(\xi_i \frac{d\eta_i}{dt} - \eta_i \frac{d\xi_i}{dt}\right) = \sum_{i=1}^{n} m_i r_i^2 \frac{d\theta_i}{dt} = c, \\ &r_i = \sqrt{\xi_i^2 + \eta_i^2}. \end{aligned} \right.$$

In case the n bodies remain collinear the line of the resultant acceleration to which each one is subject always passes through the origin. Therefore in collinear solutions we have, by the law of areas, for each body separately

$$m_i\, r_i^2\, \frac{d\theta_i}{dt} = c_i\,, \quad (i = 1, \cdots, n).$$

But when the bodies remain collinear we have also

$$\frac{d\theta_1}{dt} = \frac{d\theta_2}{dt} = \cdots = \frac{d\theta_n}{dt},$$

from which it follows that

$$\text{(3)} \qquad \frac{r_i}{r_j} = \sqrt{\frac{m_j\, c_i}{m_i\, c_j}} = a_{ij},$$

where the a_{ij} are constants. That is, if any collinear solutions exist the ratios of the distances of the bodies from the origin are constants, and it easily follows from this that the ratios of their mutual distances are also constants. They are therefore of the Lagrangian type.

If the n masses remain collinear, the ratios of their coördinates are constants, being equal to the ratios of their distances from the origin. Therefore in all collinear solutions we have

$$\text{(4)} \qquad \xi_i = x_i \xi, \quad \eta_i = x_i \eta, \quad (i = 1, \cdots, n),$$

where the x_i are constants. Substituting in equations (1) we have, as necessary conditions for the existence of the collinear solutions,

$$\text{(5)} \quad \begin{cases} \dfrac{d^2\xi}{dt^2} = -\displaystyle\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}}\, \frac{\xi}{r^3}, \quad j \neq i, \ (i = 1, \cdots, n), \\[2ex] \dfrac{d^2\eta}{dt^2} = -\displaystyle\sum_{j=1}^{n} \frac{m_j(x_i - x_j)}{x_i[(x_i - x_j)^2]^{3/2}}\, \frac{\eta}{r^3}, \\[2ex] r = \sqrt{\xi^2 + \eta^2}. \end{cases}$$

In order that ξ and η as defined by their initial values and equations (5) shall be the same for all values of i, the coefficients of ξ/r^3 and η/r^3 must be set equal to a constant independent of i. Letting $-\omega^2$ represent this constant and $r_{ij} = \sqrt{(x_i - x_j)^2}$, these conditions, which are sufficient as well as necessary for the existence of the collinear solutions, become

$$(6)\quad \begin{cases} 0 + \dfrac{m_2(x_1-x_2)}{r_{12}^3} + \dfrac{m_3(x_1-x_3)}{r_{13}^3} + \cdots + \dfrac{m_n(x_1-x_n)}{r_{1n}^3} = \omega^2 x_1, \\ \dfrac{m_1(x_2-x_1)}{r_{21}^3} + 0 + \dfrac{m_3(x_2-x_3)}{r_{23}^3} + \cdots + \dfrac{m_n(x_2-x_n)}{r_{2n}^3} = \omega^2 x_2, \\ \quad\cdot \qquad\qquad\qquad\qquad \cdot \qquad\qquad\qquad\qquad \cdot \\ \quad\cdot \qquad\qquad\qquad\qquad \cdot \qquad\qquad\qquad\qquad \cdot \\ \quad\cdot \qquad\qquad\qquad\qquad \cdot \qquad\qquad\qquad\qquad \cdot \\ \dfrac{m_1(x_n-x_1)}{r_{n1}^3} + \dfrac{m_2(x_n-x_2)}{r_{n2}^3} + \dfrac{m_3(x_n-x_3)}{r_{n3}^3} + \cdots + 0 = \omega^2 x_n. \end{cases}$$

It will be shown that ω^2 must be positive in order that real solutions of these equations shall exist. Suppose the notation is chosen so that in any solution $x_1 < \cdots < x_n$. Then the terms of the left member of the last equation are all positive. Therefore $\omega^2 > 0$ unless $x_n < 0$, which is impossible since the center of gravity relation $m_1 \xi_1 + \cdots + m_n \xi_n = 0$ holds true.

For every set of real values of $x_1, \cdots, x_n$ satisfying equations (6) the solutions of (5) are the same for all i, and these solutions substituted in (4) give the coördinates in the collinear configurations.

Since equations (5) have the same form as the differential equations in the two-body problem, it follows that *in the collinear solutions the orbits are similar conic sections.* In case the orbits are ellipses the coefficient of $-\xi/r^3$ and $-\eta/r^3$ is the product of the cube of the major semi-axis of the orbit and the square of the mean angular speed of revolution. If the undetermined scale factor be chosen so that x_i is the major semi-axis of the orbit of m_i, then the mean angular velocity of revolution of the system is ω.

The hypothesis is made that ω^2 and $m_1, \cdots, m_n$ are real positive numbers, and the problem is to find the number of real solutions of (6) for any value of n. For each of these solutions there is a six-fold infinity of collinear configurations, the six arbitrary parameters being the two which define the plane of motion, the one which defines the orientation of the orbits in their plane, the one which determines the epochs at which the bodies pass their apses, the one which determines the scale of the system, and finally the eccentricity of the orbits.

3. Outline of the Method of Solution. The solution involves a mathematical induction and consists of the following steps:

Assumption (A). To get the induction, it is assumed that for $n = \nu$ the number of real solutions of (6) for $x_1, \cdots, x_\nu$ is N_ν, whatever real positive values ω^2 and $m_1, \cdots, m_\nu$ may have. It is known from the work of Lagrange that when $\nu = 3$ we have $N_3 = 3 = \frac{1}{2}\, 3\,!$.

Theorem (B). If to the system $m_1, \cdots, m_\nu$ of positive masses an infinitesimal mass* $m_{\nu+1}$ be added, then the whole number of real solutions is

$$(\nu + 1)\, N_\nu.$$

Theorem (C). As the infinitesimal mass $m_{\nu+1}$ increases continuously to any finite positive value whatever, the total number of real solutions remains precisely $(\nu + 1)\, N_\nu$.

Conclusion (D). By successive applications of theorems (B) and (C) it is seen that the number of real solutions of (6) for $n + \nu + \mu$ is

$$N_{\nu+\mu} = (\nu + \mu)(\nu + \mu - 1) \cdots (\nu + 2)(\nu + 1)\, N_\nu.$$

When $\nu = 3$ it is known that $N_3 = \frac{1}{2} 3\,!$. Therefore

$$N_{3+\mu} = \tfrac{1}{2}(\mu + 3)\,!.$$

Let $\mu + 3 = n$ and we have

$$N_n = \tfrac{1}{2}\, n\,!. \tag{7}$$

To complete the demonstration of this conclusion it remains only to prove theorems (B) and (C).

4. Proof of Theorem (B). When there are ν finite bodies $m_1, \cdots, m_\nu$ and the infinitesimal body $m_{\nu+1}$, equations (6) become

---

* The "infinitesimal body" is in celestial mechanics a well established definite thing. In the first place it is absolutely zero. But obviously if this were all it would have no interest. It is used in association with certain differential equations which are derived in writing them first for all the masses real and positive, then letting one of them (the infinitesimal mass) approach zero. The limit of this variable mass, zero, is the infinitesimal mass, and as this mass approaches zero, its motion satisfies the differential equations, which at the limit zero for the variable mass define the motion of the infinitesimal body. Since in this limiting process the differential equations remain determinate and in all respects regular the process is fully justified. It is precisely in this sense that the infinitesimal mass is used in Theorem (B); the equations used are the limits of equations for all masses finite as $m_{\nu+1}$ approaches 0. In Theorem (C) the mass $m_{\nu+1}$ increases continuously and we follow the roots of the algebraic equations, at least as to their character.

$$\begin{cases}
\phi_1 \equiv -\omega^2 x_1 + \quad 0 \quad + \dfrac{m_2(x_1 - x_2)}{r_{12}^3} + \cdots + \dfrac{m_\nu(x_1 - x_\nu)}{r_{1\nu}^3} + \dfrac{m_{\nu+1}(x_1 - x_{\nu+1})}{r_{3\nu+1}} = 0, \\
\phi_2 \equiv -\omega^2 x_2 + \dfrac{m_1(x_2 - x_1)}{r_{12}^3} + \quad 0 \quad + \cdots + \dfrac{m_\nu(x_2 - x_\nu)}{r_{2\nu}^3} + \dfrac{m_{\nu+1}(x_2 - x_{\nu+1})}{r_{2\nu+1}^3} = 0, \\
\cdot \\
\cdot \\
\cdot \\
\phi_\nu \equiv -\omega^2 x_\nu + \dfrac{m_1(x_\nu - x_1)}{r_{1\nu}^3} + \dfrac{m_2(x_\nu - x_2)}{r_{2\nu}^3} + \cdots + \quad 0 \quad + \dfrac{m_{\nu+1}(x_\nu - x_{\nu+1})}{r_{\nu\nu+1}^3} = 0, \\
\phi_{\nu+1} \equiv -\omega^2 x_{\nu+1} + \dfrac{m_1(x_{\nu+1} - x_1)}{r_{1\nu+1}^3} + \dfrac{m_2\,(x_{\nu+1} - x_2)}{r_{2\nu+1}^3} + \cdots + \dfrac{m_\nu(x_{\nu+1} - x_\nu)}{r_{\nu\nu+1}^3} + \ 0 \ = 0.
\end{cases}$$

The last column of these equations is zero because $m_{\nu+1} = 0$, but is written for use in the proof of Theorem (C). Consequently the first ν equations, which involve $x_1, \cdots, x_\nu$ alone as unknowns, are the equations defining the solutions when $n = \nu$. By (A) there are N_ν real solutions of these equations. Let any one of these solutions be $x_1 = x_1^{(o)}, \cdots, x_\nu = x_\nu^{(o)}$. Then the last equation of (8) becomes

$$\phi_{\nu+1} \equiv -\omega^2 x_{\nu+1} + \frac{m_1(x_{\nu+1} - x_1^{(o)})}{r_{1\nu+1}^3} + \frac{m_2(x_{\nu+1} - x_2^{(o)})}{r_{2\nu+1}^3} + \cdots + \frac{m_\nu(x_{\nu+1} - x_\nu^{(o)})}{r_{\nu\nu+1}^3} = 0.$$

The number of real solutions of this equation is required.

Consider $\phi_{\nu+1}$ as a function of $x_{\nu+1}$. It is easily verified that

$$(10) \quad \begin{cases}
\phi_{\nu+1}(+\infty) = -\infty, \\
\lim\limits_{\epsilon=0} \phi_{\nu+1}(x_j^{(o)} + \epsilon) = +\infty, \quad (j = 1, \cdots, \nu), \\
\lim\limits_{\epsilon=0} \phi_{\nu+1}(x_j^{(o)} - \epsilon) = -\infty, \quad (j = 1, \cdots, \nu), \\
\phi_{\nu+1}(-\infty) = +\infty.
\end{cases}$$

Since $\phi_{\nu+1}$ is finite and continuous except at $x_{\nu+1} = x_1^{(o)}, \cdots, x_{\nu+1} = x_\nu^{(o)}$, $+\infty$, $-\infty$, it follows that there is an odd number of real solutions in each of the intervals $-\infty$ to $x_p^{(o)}$, where $x_p^{(o)}$ is the smallest $x_j^{(o)}$; $x_k^{(o)}$ to $x_l^{(o)}$, where $x_k^{(o)}$ and $x_l^{(o)}$ are any two $x_j^{(o)}$ which are adjacent; and $x_q^{(o)}$ to $+\infty$, where $x_q^{(o)}$ is the largest $x_j^{(o)}$.

But we find from (9) that

$$\frac{\partial \phi_{\nu+1}}{\partial x_{\nu+1}} \equiv -\omega^2 - \frac{2\,m_1}{r^3_{1\,\nu+1}} - \frac{2\,m_2}{r^3_{2\,\nu+1}} - \cdots - \frac{2\,m_\nu}{r^3_{\nu\,\nu+1}},$$

which is negative except at $x_{\nu+1} = x_1, \cdots, x_{\nu+1} = x_\nu$, where it is infinite. Therefore $\phi_{\nu+1}$ is a decreasing monotonic function in each of the intervals, and consequently vanishes once, and but once, in each of them. Since there are $\nu + 1$ of these intervals there are, for each real solution of the first ν equations of (8), precisely $\nu + 1$ real solutions of the last equation of (8). Since the first ν equations have, by (A), N_ν real solutions, equations (8) altogether have precisely $(\nu + 1)\ N_\nu$ real solutions. This completes the demonstration of Theorem (B).

4. Proof of Theorem (C). Let $x_j = x_j^{(0)}$ $(j = 1, \cdots, \nu + 1)$ be any one of the $(\nu + 1) N_\nu$ real solutions of equations (8) which are known to exist for $m_{\nu+1} = 0$. It will be shown that as $m_{\nu+1}$ increases continuously to any finite positive quantity whatever, the $x_j^{(0)}$ may be made to vary continuously so as always to satisfy equations (8), and that during these variations the $x_j^{(0)}$ remain distinct, finite, and real. From this it will follow that there are at least $(\nu + 1)\ N_\nu$ real solutions of (8) for every set of finite positive values of $m_1, \cdots, m_{\nu+1}$. It will also be shown that no new solutions can appear as $m_{\nu+1}$ increases from zero to any finite value. Hence it will follow that the number of real solutions of (8) is exactly $(\nu + 1)\ N_\nu$ for all finite positive values of $m_1, \cdots, m_{\nu+1}$.

The roots of algebraic equations are continuous functions of the coefficients so long as the roots are finite and the equations do not have indeterminate forms. Consequently the $x_j^{(0)}$ are continuous functions of $m_{\nu+1}$ if no $x_i^{(0)}$ becomes infinite and if no $x_i^{(0)} = x_j^{(0)}$. The real roots of algebraic equations having real coefficients can disappear only by passing to infinity, or by an even number of real solutions becoming conjugate complex quantities in pairs. Therefore we have to determine (1) whether any finite $x_i^{(0)}$ can become equal to any $x_j^{(0)}$, (2) whether any $x_i^{(0)}$ can become infinite, and (3) whether any two real solutions can become conjugate complex quantities for any finite positive values of $m_1, \cdots, m_{\nu+1}$.

(1). The masses $m_1, \cdots, m_{\nu+1}$ are by hypothesis all positive. Let the notation be chosen so that for any values of $m_1, \cdots, m_{\nu+1}$ for which the $x_i^{(0)}$ are all distinct we have the inequalities $x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}$. Let us suppose that as some mass is changed we find $x_j^{(0)} - x_i^{(0)}$ approaching zero in such a way that $x_j^{(0)}$ and $x_i^{(0)}$ remain finite. That is,

r_{ij}, which occurs only in the expressions for ϕ_i and ϕ_j, approaches zero. Suppose $i < j$. Then the term involving r_{ij} becomes negatively infinite in ϕ_i and positively infinite in ϕ_j. Consider $\phi_i = 0$. Another r_{ik} must approach zero in order to restore the finite value of the function ϕ_i, and the term involving r_{ik} must become positively infinite as r_{ik} approaches zero. Therefore we see from (8) that $k < i$. But r_{ik} enters besides only in ϕ_k, and similar reasoning shows that r_{kl}, where $l < k$, must also approach zero. In this manner we are driven to the conclusion finally that an r_{pq}, where one of the subscripts is unity, approaches zero. Then consider $\phi_1 = 0$. All its terms except $-\omega^2 x_1$ are negative, and since one of its r_{ij}, that is r_{pq}, approaches zero, the first equation of (8) can not be satisfied. Consequently the original assumption that some r_{ij} can approach zero for finite values of $x_1^{(0)}, \cdots, x_{\nu+1}^{(0)}$ and finite *positive* values of $m_1, \cdots, m_{\nu+1}$ leads to an impossibility, and is therefore false.

(2). Multiplying equations (8) by $m_1, m_2, \cdots, m_{\nu+1}$ respectively and adding, we get

$$-\omega^2(m_1 x_1 + m_2 x_2 + \cdots + m_\nu x_\nu + m_{\nu+1} x_{\nu+1}) = 0.$$

It follows from this equation that no $x_i^{(0)}$ alone can become infinite, and that if one becomes positively infinite, then some other one must become negatively infinite.

Suppose the notation is again chosen so that $x_1^{(0)} < x_2^{(0)} < \cdots < x_\nu^{(0)} < x_{\nu+1}^{(0)}$. Then if any $x_i^{(0)}$ becomes negatively infinite $x_1^{(0)}$ must do the same, and from the equation above it follows that $x_{\nu+1}^{(0)}$ must become positively infinite. Now suppose this to occur and consider the equation $\phi_1 = 0$. In order that this equation may remain satisfied, $x_2^{(0)}$ must also become negatively infinite in such a way that $x_1^{(0)} - x_2^{(0)}$ shall approach zero. But now it follows from $\phi_2 = 0$, since $-\omega^2 x_2^{(0)}$ and $\dfrac{m_1(x_2^{(0)} - x_1^{(0)})}{r_{12}^3}$ are both positive, that $x_3^{(0)}$ must also become negatively infinite in such a way that $x_2^{(0)} - x_3^{(0)}$ shall approach zero. It follows similarly from $\phi_3 = 0$ that $x_4^{(0)}$ must become negatively infinite in such a way that $x_3^{(0)} - x_4^{(0)}$ shall approach zero. This reasoning continues until it is found that $x_1^{(0)}, \cdots, x_{\nu+1}^{(0)}$ must all become negatively infinite. But $x_{\nu+1}^{(0)}$ at least must become positively infinite, and we are thus led to a contradiction. Similarly it can be proved that no $x_i^{(0)}$ can become positively infinite.

In order to prove now that as $m_{\nu+1}$ approaches zero, equations (8) remain determinate and their solutions finite, and that there are accordingly no others besides those obtained in Theorem (B), consider a solution $x_1, \cdots, x_{\nu+1}$ in which the x_j will be distinct for a set of positive values of $m_1, \cdots, m_{\nu+1}$, and then let m_i approach zero as a limit.

In the first place, if x_i approaches neither x_{i-1} nor x_{i+1} as a finite limit as m_i approaches zero as a limit, then by the reasoning of (1) and (2) above no x_j can approach any x_k as a limit.

In the second place, x_i can not approach x_{i+1} as a finite limit as m_i approaches zero unless x_{i-1} also approaches x_{i+1}, for otherwise $\phi_i = 0$ can not be satisfied. But if x_{i-1} approaches x_{i+1} as a limit as m_i approaches zero, then $\phi_{i-1} = 0$ and $\phi_{i+1} = 0$ can not be satisfied unless x_{i-2} and x_{i+2} also approach x_{i+1} as a limit. This shifts the difficulty to $\phi_{i-2} = 0$ and $\phi_{i+2} = 0$, and so on until $\phi_1 = 0$ and $\phi_{\nu+1} = 0$ are reached, which can not be satisfied under the hypothesis.

In the third place, x_i can not become positively infinite as m_i approaches zero, for then $\phi_i = 0$ can not be satisfied unless x_{i-1} becomes infinite in such a way that $x_i - x_{i-1}$ approaches zero. Continuing through $\phi_{i-1} = 0$, etc., we are led to the conclusion that $x_1, \cdots, x_{\nu+1}$ all become positively infinite; but then the center of gravity equation can not be satisfied. In a similar manner it can be proved that as m_i approaches zero as a limit no x_j can approach any x_k or become infinite. Consequently the solutions all remain regular as m_i approaches zero as a limit.

(3). Since the solutions of (8) are continuous functions of $m_{\nu+1}$, it follows that no two solutions which are real for $m_{\nu+1} = 0$ can ever become conjugate complex solutions for any real value of $m_{\nu+1}$ without having first become equal; and similarly, no two solutions which are complex for $m_{\nu+1} = 0$ can ever become real for any real value of $m_{\nu+1}$ without having first become equal. Consequently if a multiple solution of (8) is impossible for every set of finite positive values of $m_1, \cdots, m_{\nu+1}$, it is impossible that any real solutions should disappear by becoming imaginary, or that any imaginary solutions should become real.

The conditions that $x = x^{(0)}$ shall be a multiple solution of $f(x) = 0$ are that $f(x^{(0)}) = 0$ and $f'(x^{(0)}) = 0$. The corresponding conditions that a set of simultaneous algebraic equations shall have a multiple solution are that a set of values of the variables shall satisfy the equations and that the Jacobian of the functions with respect to the dependent variables shall vanish for the same set of values. That is, the conditions that $x_j = x_j^{(0)}$,

$(j = 1, \cdots, \nu + 1)$, shall be a multiple solution of (8) are that these values shall satisfy (8) and also the equation

$$(11) \qquad \Delta \equiv \begin{vmatrix} \frac{\partial \phi_1}{\partial x_1} & \frac{\partial \phi_1}{\partial x_2} & \cdots & \frac{\partial \phi_1}{\partial x_{\nu+1}} \\ \frac{\partial \phi_2}{\partial x_1} & \frac{\partial \phi_2}{\partial x_2} & \cdots & \frac{\partial \phi_2}{\partial x_{\nu+1}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{\partial \phi_{\nu+1}}{\partial x_1} & \frac{\partial \phi_{\nu+1}}{\partial x_2} & \cdots & \frac{\partial \phi_{\nu+1}}{\partial x_{\nu+1}} \end{vmatrix} = 0.$$

Consider two solutions of a set of algebraic equations having real coefficients. If they change from real to conjugate complex quantities or from conjugate complex to real quantities for a certain value of a continuously varying parameter, then for this value of the parameter, they are not only equal, but they are also *real*. Consequently we shall have occasion to examine Δ only when all of its elements are real. We shall show that it cannot vanish for any finite set of real values of the x_j when $m_1, \cdots, m_{\nu+1}$ are positive, and consequently that it can not vanish for any particular set which satisfies equations (8). When this is established we shall have proved that all the solutions of (8) which are real for $m_{\nu+1} = 0$ remain real when $m_{\nu+1}$ increases to any positive value, and those which are complex remain complex.

From equations (8) and (11) we find

$$(12) \qquad \Delta \equiv \begin{vmatrix} a_1 & \frac{2m_2}{r^3_{12}} & \cdots & \frac{2m_{\nu+1}}{r^3_{1\,\nu+1}} \\ \frac{2m_1}{r^3_{21}} & a_2 & \cdots & \frac{2m_{\nu+1}}{r^3_{2\,\nu+1}} \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \frac{2m_1}{r^3_{\nu+1\,1}} & \frac{2m_2}{r^2_{\nu+1\,2}} & \cdots & a_{\nu+1} \end{vmatrix}$$

where

$$
\begin{aligned}
a_1 &= -\omega^2 - 0 - \frac{2m_2}{r_{12}^3} - \cdots - \frac{2m_{\nu+1}}{r_{1\nu+1}^3}, \\
a_2 &= -\omega^2 - \frac{2m_1}{r_{21}^3} - 0 - \frac{2m_3}{r_{23}^3} - \cdots - \frac{2m_{\nu+1}}{r_{2\nu+1}^3}, \\
&\cdot \\
&\cdot \\
&\cdot \\
a_{\nu+1} &= -\omega^2 - \frac{2m_1}{r_{\nu+11}^3} - \cdots - \frac{2m_\nu}{r_{\nu+1\nu}^3} - 0.
\end{aligned}
$$

If $m_{\nu+1} = 0$ this determinant breaks up into the product of a determinant of the same type as (12) when all the m_i are distinct from zero and a factor essentially negative. Therefore it is sufficient to consider the general case in which all the m_i are positive in examining whether Δ can vanish or not.

Several properties of Δ are evident. (a) If the ith row be multiplied by m_i ($i = 1, \cdots, \nu + 1$), the determinant becomes symmetrical. (b) The sum of the elements in each row is $-\omega^2$, from which it follows that the expansion of the determinant contains ω^2 as a factor. (c) The expansion of the determinant contains $(-1)^{\nu+1}\omega^{2(\nu+1)}$ as one of its terms, and since all the m_i are positive and all the x_i are real the sign of all the terms coming from the product of the elements of the main diagonal is $(-1)^{\nu+1}$.

When Δ is completely expanded all those terms not having the sign $(-1)^{\nu+1}$ are cancelled by terms coming from the product of the main diagonal elements, and since the term $(-1)^{\nu+1}\omega^{2(\nu+1)}$ is certainly present the determinant can in no case be zero. The following demonstration of this fact was invented in 1907 by Mr. T. H. Hildebrandt, now of the University of Michigan, as a class exercise.*

Since the determinant contains ω^2 as a factor every term in its expansion must depend upon at least one of the elements of the main diagonal. Fasten the attention upon any term of the expansion. It can be supposed without loss of generality that it depends upon the first main diagonal element. In the expansion of the determinant this element is multiplied by its minor; consequently we must see if the minor can vanish. The minor is of the same form as the original determinant, and the sum of the elements of its ith row is $-\omega^2 - 2m_1/r_{1i}^3$. Consequently every term in the expansion of the minor will contain at least one of the $-\omega^2 - 2m_1/r_{1i}^3$ as a factor. But these elements appear only in the main diagonal of the minor. Hence all terms in the expansion of the minor which do not vanish depend upon at least one element of the main diagonal. In considering our particular term it may be supposed without loss of generality to depend upon the first main diagonal element of the minor. In the expansion of the original determinant the product of these two diagonal elements will be multiplied by the co-factor of the minor of the second order of which they are the main diagonal. This co-factor has properties similar to those of the first minor just considered, and in the same way we prove that at least one of its diagonal elements must be involved in the term in question. That is, the term under consideration depends upon at least three elements of the main diagonal. Continuing in this manner we prove that any term in the final expansion different from zero depends upon all the elements of the main diagonal which are all of the same sign in every one of their terms. Consequently all the terms which do not cancel out in the expansion of the determinant have the sign $(-1)^{\nu+1}$. And we have seen that there is at least one such term, viz., $(-1)^{\nu+1}\omega^{2(\nu+1)}$. Therefore the determinant can not only never vanish, but it can never become less than $\omega^{2(\nu+1)}$ in numerical value.

---

* An earlier proof was devised by the author, and still another jointly by Professor N. B. MacLean, of the University of Manitoba, and Mr. E. J. Moulton, now of the University of Wisconsin.

Since Δ can never vanish for real finite $x_j^{(o)}$ when all the m_j are real and positive, it follows that no real solutions can ever be lost or gained as the m_j vary, and therefore that the number of real solutions of (8) is always exactly $(\nu + 1)\, N_\nu = \frac{1}{2}\,(\nu + 1)\,!$.

5. Computation of the Solutions of Equations (6). There are well known methods of finding the roots of a single numerical algebraic equation of high degree, but they are not readily applicable to simultaneous equations of high degree. However, when the order of the masses has been chosen, equations (6) will become polynomials in $x_1, \cdots, x_n$ after they have been cleared of fractions. Then by rational processes $n - 1$ of the x_i can be eliminated from these equations giving a single equation in the remaining unknown. The solutions of this equation can be found by the usual methods, and the results can be used to eliminate one unknown. By repeated applications of this process to the successively reduced equations, the solutions can all be found. The single one satisfying the conditions of reality of $x_1, \cdots, x_n$ and their order relations is the one desired.

The solutions can also be found by a method closely related to that by means of which their existence was proved above. Suppose for $m_i = m_i^{(0)}$, $(i = 1, \cdots, n)$, a solution $x_i = x_i^{(0)}$ of equations (6) is known. The $m_i^{(0)}$ are supposed to be zero or positive. Suppose it is desired to find the corresponding solution, that is, the one in which the masses are arranged on the line in the same order, for $m_i = m_i^{(0)} + \mu_i$. Let the corresponding set of the x_i satisfying (6) be $x_i = x_i^{(0)} + \xi_i$, where the ξ_i are functions of $\mu_1, \cdots, \mu_n$ to be determined. Substituting $x_i = x_i^{(0)} + \xi_i$ and $m_i = m_i^{(0)} + \mu_i$ in (6), making use of the notation of (8), expanding as power series in the ξ_i and μ_i (which is always possible since we have shown no $x_i^{(0)}$ can become infinite and no $x_i^{(0)}$ can equal any $x_j^{(0)}$), and remembering that $x_i = x_i^{(0)}$ is a solution of (6) for $m_i = m_i^{(0)}$, we have

$$(13)\quad \left\{\begin{array}{l} \displaystyle\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial m_j}\mu_j, \\ \qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ \qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ \qquad\cdot \qquad\qquad\qquad\qquad\qquad\qquad\qquad \cdot \\ \displaystyle\sum_{j=1}^{n} \frac{\partial \phi_n}{\partial x_j}\xi_j + \sum_{i=2}^{\infty} \frac{1}{i!}\left[\sum_{j=1}^{n} \frac{\partial \phi_1}{\partial x_j}\xi_j\right]^i = -\sum_{j=1}^{n} \frac{\partial \phi_n}{\partial m_j}\mu_j, \end{array}\right.$$

where $\left[\sum_{j=1}^{n} \frac{\partial \phi_k}{\partial x_j}\xi_j\right]^i$ are the symbolic powers used in connection with the power series expansions of functions of several variables.

The determinant of the terms of the first degree in the ξ_j in equations (13) is the Δ of equation (11) which we have proved can never vanish in this problem. Therefore equations (13) can be uniquely solved for $\xi_1, \cdots, \xi_n$ as power series in $\mu_1, \cdots, \mu_n$ by the method of undetermined coefficients, and these series will converge for $|\mu_i| > 0$ but sufficiently small. Suppose they converge if $|\mu_i| \leqq r$. Keeping the μ_i within this limit a solution $x_i = x_i^{(1)}$ is computed. Then this can be used as a starting point for a second application of the process, which can be repeated as many times as may be desired.

Hence, to find the solution in which the bodies $m_1, \cdots, m_n$ have any finite positive values and lie in a determined order on the line, we may start with m_1, m_2, and m_3 and solve the Lagrangian quintic* which defines their distribution on the line. Then we add an infinitesimal body m_4 and find its position by solving (9) in which now $\nu = 3$. Then we increase the infinitesimal mass, step by step, to the required finite value, and compute the corresponding $x_1, \cdots, x_4$. Making use of the fact that for $|\xi_i| < \rho$ we have for all values of i

---

* Tisserand's *Mécanique Céleste*. vol. 1, p. 155, or Moulton's *Introduction to Celestial Mechanics*, p. 216.

$$|\phi_i| < A, \qquad\qquad \Delta \geqq \omega^{2n},$$

where A is a number depending on ρ, ω^2, $m_1, \cdots, m_n$, it can be proved that by this process any finite value m_4 can be reached in a finite number of steps. After the required value of m_4 has been reached the process can be repeated for m_5, etc., to any finite number of bodies. Notwithstanding the fact that this would be very laborious if the number of bodies was large, yet we must regard the problem as completely solved both theoretically and practically.

II. Determination of the Masses when the Positions are Given.

6. Determination of the Masses when n is Even. Suppose ω^2 and the n distinct points on a line, $x_1, \cdots, x_n$, are given, and consider the problem of determining $m_1, \cdots, m_n$ so that the straight line solutions shall exist. There will be no loss of generality in selecting the notation so that $x_1 < x_2 < \cdots < x_n$. With this choice of notation equations (6), which are the necessary and sufficient conditions for the solutions, become

$$(14)\quad \left\{\begin{array}{llllll} 0 & + \dfrac{m_2}{r_{12}^2} + \dfrac{m_3}{r_{13}^2} & + \cdots + & \dfrac{m_{n-1}}{r_{1\,n-1}^2} & + \dfrac{m_n}{r_{1n}^2} & = -\omega^2 x_1, \\ -\dfrac{m_1}{r_{21}^2} & + 0 \quad + \dfrac{m_3}{r_{23}^2} & + \cdots + & \dfrac{m_{n-1}}{r_{2\,n-1}^2} & + \dfrac{m_n}{r_{2n}^2} & = -\omega^2 x_2, \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ -\dfrac{m_1}{r_{n-1\,n}^2} & - \dfrac{m_2}{r_{n-1\,2}^2} - \dfrac{m_3}{r_{n-1\,3}^2} & - \cdots - & 0 & + \dfrac{m_n}{r_{n-1\,n}^2} & = -\omega^2 x_{n-1}, \\ -\dfrac{m_1}{r_{n1}^2} & - \dfrac{m_2}{r_{n2}^2} - \dfrac{m_3}{r_{n3}^2} & - \cdots - & \dfrac{m_{n-1}}{r_{n\,n-1}^2} & + 0 & = -\omega^2 x_n. \end{array}\right.$$

The m_i enter these equations linearly and are therefore uniquely determined if the determinant

$$
(15) \qquad D = \begin{vmatrix}
0 & \frac{1}{r^2_{12}} & \frac{1}{r^2_{13}} & \cdots & \frac{1}{r^2_{1\,n-1}} & \frac{1}{r^2_{1n}} \\
\frac{-1}{r^2_{21}} & 0 & \frac{1}{r^2_{23}} & \cdots & \frac{1}{r^2_{2\,n-1}} & \frac{1}{r^2_{2n}} \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\cdot & & & & & \cdot \\
\frac{-1}{r^2_{n-1\,1}} & \frac{-1}{r^2_{n-1\,2}} & \frac{-1}{r^2_{n-1\,3}} & \cdots & 0 & \frac{1}{r^2_{n-1\,n}} \\
\frac{-1}{r^2_{n1}} & \frac{-1}{r^2_{n2}} & \frac{-1}{r^2_{n3}} & \cdots & \frac{-1}{r^2_{n\,n+1}} & 0
\end{vmatrix}
$$

is distinct from zero. This is a skew-symmetric determinant, and when n is even it is the square of an associated Pfaffian, and, therefore, is not in general zero. Therefore if n is even the masses are in general uniquely determined when ω^2 and $x_1, \cdots, x_n$ are given, though it should be noted that they are not necessarily all positive. When they are negative they have no physical interpretation.

7. Determination of the Masses when n is odd. In this case the skew-symmetric determinant is identically zero, but its first minors of the main diagonal elements, being skew-symmetrical determinants of even order, are in general all distinct from zero; consequently the x_i must satisfy one relation in order that equations (14) shall be consistent. To get this relation add the equation

$$m_1 x_1 + m_2 x_2 + \cdots + m_n\, x_n = 0,$$

which is a consequence of (14), to the set of equations. In order that they shall be consistent their eliminant

$$E \equiv \begin{vmatrix} 0 & x_1 & x_2 & \cdots & x_n \\ -x_1 & 0 & \dfrac{1}{r_{21}^2} & \cdots & \dfrac{1}{r_{n1}^2} \\ -x_2 & -\dfrac{1}{r_{12}^2} & 0 & \cdots & \dfrac{1}{r_{n2}^2} \\ \vdots & & & & \vdots \\ -x_n & -\dfrac{1}{r_{1n}^2} & -\dfrac{1}{r_{2n}^2} & \cdots & 0 \end{vmatrix}$$

must vanish. This is also a skew-symmetrical determinant and is the square of the Pfaffian

$$F \equiv \left| \begin{array}{cccc} x_1 & x_2 & \cdots & x_n \\ & \dfrac{1}{r_{12}^2} & \cdots & \dfrac{1}{r_{1n}^2} \\ & & & \vdots \\ & & & \dfrac{1}{r_{n\,n-1}^2} \end{array} \right| = 0. \tag{16}$$

Equation (16) can be found also by solving any $n-1$ equations of (14) for the corresponding m_i and substituting the solutions in the remaining one. The result is a sum of determinants which can be shown to be the expansion of F multiplied by the square root of the determinant of the coefficients of the $n-1$ m_i in the equations used.

When $F=0$ is satisfied by $x_1, \cdots, x_n$, equations (14) are consistent. Then, after any m_i has been chosen arbitrarily, the corresponding $n-1$ equations can in general be solved uniquely for the remaining m_j, and the unused equation will be satisfied because $F=0$.

8. Discussion of Case $n = 3$. When $n=3$ the determinant D becomes

$$D \equiv \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} - \frac{1}{(r_{12}\, r_{23}\, r_{13})^2} \equiv 0,$$

and the Pfaffian F is

$$F \equiv \frac{x_1}{r_{23}^2} - \frac{x_2}{r_{13}^2} + \frac{x_3}{r_{12}^2} = 0. \tag{17}$$

It will now be shown that when any two of x_1, x_2, x_3 are chosen so as to satisfy the conditions $x_1 < x_2 < x_3$, the third is uniquely determined by (17) and these inequalities. From the fact that in this case $r_{13} > r_{12}$, $r_{13} > r_{23}$ it follows that if x_2 is positive then $-x_2/r_{13}^2 + x_3/r_{12}^2$ is positive, and therefore that x_1 must be negative in order that (17) may be satisfied. If x_2 is negative x_1, being less, must also be negative. That is, x_1 is necessarily negative, and similarly x_3 is necessarily positive.

Suppose x_2 and x_3 are chosen and consider F as a function of x_1. Then we have at once

$$(18)\qquad \lim_{x_1 \doteq -\infty} F(x_1) = -\infty\,, \quad \lim_{\epsilon \doteq 0} F(x_2 - \epsilon) = +\infty\,, \quad \frac{\partial F}{\partial x_1} = \frac{1}{r_{23}^2} - \frac{2x_2}{r_{13}^3} + \frac{2x_3}{r_{12}^3}.$$

From the inequalities $x_2 < x_3$ and $r_{12} < r_{13}$ it follows that $\dfrac{\partial F}{\partial x_1}$ is positive for $x_1 < x_2$. Therefore there is one and but one solution of (17) for $x_1 < x_2$ when x_2 and a positive x_3 are chosen. By symmetry there is but one solution of (17) for $x_3 > x_2$ when a negative x_1 and x_2 are chosen.

When x_1 and x_3 are chosen respectively negative and positive but otherwise arbitrarily we have, considering F as a function of x_2,

$$(19)\qquad \lim_{\epsilon \doteq 0} F(x_1 + \epsilon) = +\infty\,, \quad \lim_{\epsilon = 0} F(x_3 - \epsilon) = -\infty\,, \quad \frac{\partial F}{\partial x_2} = \frac{2x_1}{r_{23}^3} - \frac{1}{r_{13}^2} - \frac{2x_3}{r_{12}^3} < 0.$$

Therefore there is one and but one solution of (17) for x_2 satisfying the inequalities $x_1 < x_2 < x_3$.

Suppose a negative x_1, a positive x_3, and m_2 are given arbitrarily and that x_2 is defined by (17). From (14) we have

$$(20)\qquad \left\{\begin{aligned} m_1 &= -\, r_{13}^2 \left[\frac{m_2}{r_{23}^2} - \omega^2 x_3\right], \\ m_3 &= -\, r_{13}^2 \left[\frac{m_2}{r_{12}^2} + \omega^2 x_1\right]. \end{aligned}\right.$$

Since x_1 is negative and x_3 positive it follows from (20) that m_2 can be taken so small that m_1 and m_3 are positive. If $-x_1$ is not equal to x_3, then a positive m_2 can be determined so that m_1 and m_3 shall both be positive, one positive and one negative, or both negative; but m_1, m_2, and m_3 can not all be negative.

THE UNIVERSITY OF CHICAGO,
NOVEMBER, 1909.